Higher time regularity of SPDE-based Gaussian processes using fractional brownian motion

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OUTLINE

GEOSTATISTICAL MODELING

Geostatistical paradigm: over the spatial domain $\mathcal{D}$

Gaussian Random Field		Observed variable
$Z : \{Z(\boldsymbol{p}) : \boldsymbol{p} \in \mathcal{D}\}$	$\underset{\longrightarrow}{\text{Realization}}$	$z : \{z(\boldsymbol{p}) : \boldsymbol{p} \in \mathcal{D}\}$
High correlation		High "similarity"

- Allows to model data which are not independent, identically distributed
- **Covariance function** C_Z:

$$\begin{array}{rcccl} C_Z & : & \mathcal{D} \times \mathcal{D} & \to & \mathbb{R} \\ & & (\boldsymbol{p}_1, \boldsymbol{p}_2) & \mapsto & C_Z(\boldsymbol{p}_1, \boldsymbol{p}_2) = \mathrm{Cov}(Z(\boldsymbol{p}_1), Z(\boldsymbol{p}_2)) \end{array}$$

$\rightarrow$ used to model the spatial structure observed on the variable/data

CHALLENGES IN PRACTICE

Non-euclidean domains

- Extensive literature for the sphere: Marinucci and Peccati (2011); Lang et al. (2015); Lantuéjoul et al. (2019); Emery and Porcu (2019)

Non-stationarity

- Examples of proposed methods: Karhunen-Loève expansions (Lindgren, 2012), Space deformation models (Sampson and Guttorp, 1992), Convolution models (Higdon et al., 1999)

Big "N" problem

- Need to restrict the choice of models to work with **sparse matrices**: Compactly-supported or tapered covariance functions (Gneiting, 2002; Furrer et al., 2006), Markovian models (Rue and Held, 2005)

THE SPDE APPROACH

Basic idea: if $\mathcal{Z}$ is an isotropic Markovian field over $\mathbb{R}^d$, then it is **equivalently** characterized by (Whittle, 1954; Rozanov, 1977):

Spectral density

$$\Gamma : \boldsymbol{\xi} \in \mathbb{R}^d \mapsto \frac{1}{P(\|\boldsymbol{\xi}\|^2)}$$

Stochastic partial differential equation (SPDE)

$$\boxed{P(-\Delta)^{1/2}\mathcal{Z} = \mathcal{W}}$$

- $\mathcal{W}$: Gaussian white noise
- $P(-\Delta)^{1/2}\mathcal{Z} := \mathscr{F}^{-1}\left[\boldsymbol{\xi} \mapsto P(\|\boldsymbol{\xi}\|^2)^{1/2} \times \mathscr{F}[\mathcal{Z}](\boldsymbol{\xi})\right]$

where P is a **polynomial**, strictly positive over $\mathbb{R}_+$

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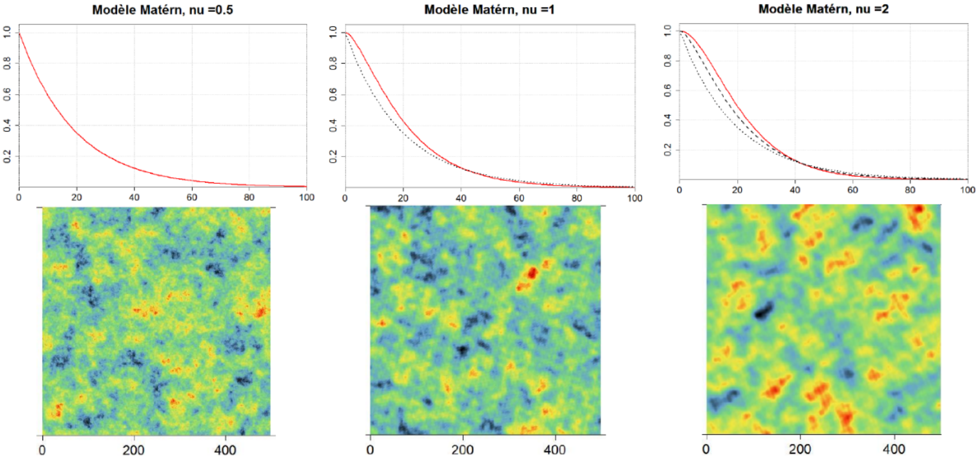
$\rightarrow$ In particular, if $P(x) = (\kappa^2 + x)^\alpha$, i.e. if we consider the SPDE

$$(\kappa^2 - \Delta)^{\alpha/2}\mathcal{Z} = \mathcal{W}$$

then Z has a Matérn covariance function

$$\mathrm{Cov}\big(\mathcal{Z}(\boldsymbol{x}+\boldsymbol{h}), \mathcal{Z}(\boldsymbol{x})\big) = C(\|h\|) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}\big(\kappa\|h\|\big)^\nu \mathcal{K}_\nu\big(\kappa\|h\|\big), \quad \nu = \alpha - d/2$$

MATÉRN RANDOM FIELDS



Simulations of Gaussian random fields with a Matérn covariance

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SPDE approach: Lindgren et al. (2011) use this last characterization of isotropic Markovian fields

Problem	Solution proposed
Non-euclidean domains, Non-stationarity	Define the SPDE on manifolds or use varying parameters

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Problem	Solution proposed
Big “N” problem	Use the finite element method to solve the SPDE

A CLASS OF RANDOM FIELDS

Let $\mathcal{L}$ be a second-order self-adjoint elliptic operator with smooth coefficients, eg.

$$\mathcal{L} = -\Delta, \quad \mathcal{L} = \kappa^2(\cdot) - \mathrm{div}(H(\cdot)\nabla)$$

- **Spectral theorem on compact Riemannian manifolds** $\mathcal{M} = (\mathcal{D}, g)$:
  - $\mathcal{L}$ has discrete eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ with smooth eigenfunctions $\{e_k : k \in \mathbb{N}\}$
  - The eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ can be taken to form an orthonormal basis of $L^2(\mathcal{M})$

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- Consider the $L^2(\mathcal{M})$-valued random variables defined by

$$\boxed{\mathcal{Z} = \sum_{k\in\mathbb{N}} \gamma(\lambda_k) W_k \; e_k} \quad \text{where } \{W_k\}_{k\in\mathbb{N}} \sim \mathrm{IID}\mathcal{N}(0,1)$$

and $\gamma : \mathbb{R}_+ \to \mathbb{R}$ such that $|\gamma(\lambda)| = \mathcal{O}_{\lambda\to\infty}(|\lambda|^{-\beta})$ with $\beta > d/4$ (eg. $\gamma(\lambda) = (\kappa^2 + x)^{-\alpha/2}$)

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- Covariance properties (Pereira, 2019): when $(\mathcal{M}, g) = ([0,1]^d, g)$ and $\mathcal{L} = -\Delta$

$$\mathrm{Cov}\left(\mathcal{Z}(\boldsymbol{p}), \mathcal{Z}(\boldsymbol{p} + d\boldsymbol{p})\right) \approx C_0\left(\sqrt{g_{\boldsymbol{p}}(d\boldsymbol{p}, d\boldsymbol{p})}\right) \quad \text{where} \quad C_0 = \mathscr{F}^{-1}[\gamma^2]$$

■ DISCRETIZING THE FIELD

$L^2(\mathcal{M})$
Self-adjoint differential operator $\mathcal{L}$

$\downarrow$

Spectral theorem: $\{(\lambda_k, e_k) : k \in \mathbb{N}\}$
eigenvalues/functions of $\mathcal{L}$

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$L^2(\mathcal{M})$-valued random variables

$$\mathcal{Z} = \sum_{k\in\mathbb{N}} \underbrace{\gamma(\lambda_k) W_k}_{\text{independent Gaussian weights}} \; e_k$$

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- ✓ General approach
- ✓ Local definition of covariance:

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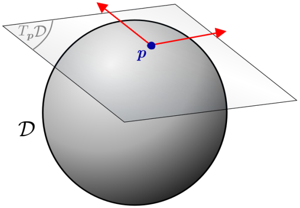

Lengths and angles
of tangent vectors $\boldsymbol{u}, \boldsymbol{v}$:

$$\|\boldsymbol{u}\|_{\boldsymbol{p}} = \sqrt{g_{\boldsymbol{p}}(\boldsymbol{u}, \boldsymbol{u})}$$

$$\cos\left(\theta(\boldsymbol{u}, \boldsymbol{v})\right) = \frac{g_{\boldsymbol{p}}(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{u}\|_{\boldsymbol{p}} \|\boldsymbol{v}\|_{\boldsymbol{p}}}$$

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FEM basis $V_{N_h} = \mathrm{span}\{\psi_1, \dots, \psi_{N_h}\}$
$\mathcal{L}_h$ = Galerkin approximation of $\mathcal{L}$

$\downarrow$

"Spectral theorem": $\left\{(\lambda_k^{(h)}, e_k^{(h)}) : k \in [\![1, N_h]\!]\right\}$
eigenvalues/functions of $\mathcal{L}_h$

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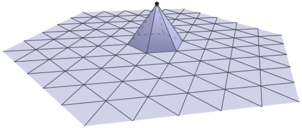
V_{N_h}-valued random variables

$$\mathcal{Z}_h = \sum_{k=1}^{N_h} \underbrace{\gamma(\lambda_k^{(h)})W_k^{(h)}}_{\text{independent Gaussian weights}} e_k^{(h)} = \sum_{i=1}^{N_h} Z_i \psi_i$$



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- ✓ Explicit computation (Lang and Pereira, 2023):

$$\boxed{\boldsymbol{Z} = \boldsymbol{C}^{-1/2}\gamma(\boldsymbol{S})\boldsymbol{W}, \quad \boldsymbol{W} \sim \mathcal{N}(0, \boldsymbol{I})}$$

where $\boldsymbol{S} = \boldsymbol{C}^{-1/2}\boldsymbol{R}\boldsymbol{C}^{-1/2}$,
$\boldsymbol{C} = [\langle\psi_i, \psi_j\rangle], \boldsymbol{R} = [\langle\mathcal{L}\psi_i, \psi_j\rangle]$

$\rightarrow$ **sparse matrices**

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FEM convergence result ⟷

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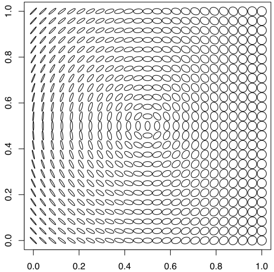
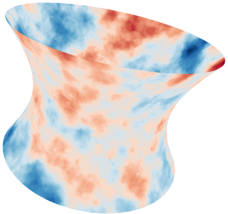
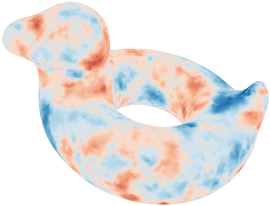
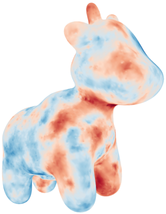
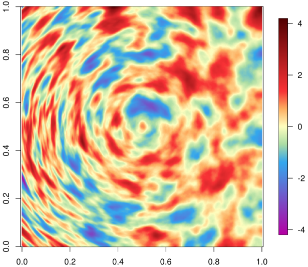
■ EXAMPLES OF SAMPLED RANDOM FIELDS

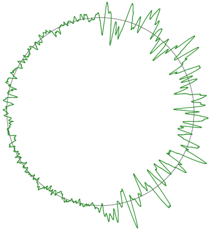


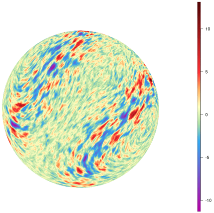












OUTLINE

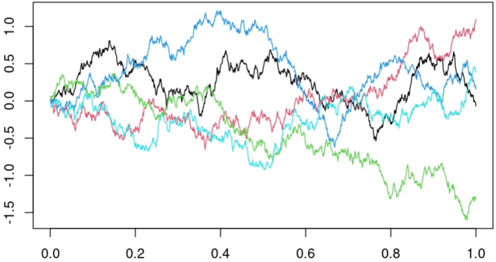
BROWNIAN MOTION AND WHITE NOISE

- A Wiener process / Brownian motion $(\beta_t)_{t\in[0,T]}$ is a (real-valued) stochastic process such that
  - $\beta_0 = 0$ and $(\beta_t)_{t\in[0,T]}$ has (almost-surely) continuous trajectories
  - For any $0 \leq t_1 < \cdots < t_n \leq T$,

$$\beta_{t_1}, \beta_{t_2} - \beta_{t_1}, \ldots, \beta_{t_{n-1}} - \beta_{t_n}$$

are independent and are Gaussian-distributed:

$$\beta_{t_{i+1}} - \beta_{t_i} \sim \mathcal{N}(0, (t_{i+1} - t_i))$$



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- If $\mathcal{W}$ is a white noise, then $(\beta_t)_{t\in[0,T]}$ defined as

$$\beta_t = \mathcal{W}([0,T]) = \int \mathbf{1}_{[0,t]} d\mathcal{W} = \int_0^t \mathcal{W}(dt)$$

  is a Brownian motion.
  $\rightarrow$ White noise can be seen as the (weak) derivative of Brownian motion

SPACE-TIME NOISE?

- Current SPDE:

$$(\kappa^2 - \Delta)^{\alpha/2}\mathcal{Z} = \mathcal{W}$$

$\rightarrow$ Space-time equivalent of white noise $\mathcal{W}$? $\rightarrow$ Q-Wiener process

- A Q-Wiener process $(\mathcal{B}_t^Q)_{t\in[0,T]}$ is a $L^2(\mathcal{D})$-valued stochastic process such that
  - $\mathcal{B}_0^Q = 0$ and $(\mathcal{B}_t^Q)_{t\in[0,T]}$ has (almost-surely) continuous trajectories
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$$\mathcal{B}_{t_1}^Q, \mathcal{B}_{t_2}^Q - \mathcal{B}_{t_1}^Q, \ldots, \mathcal{B}_{t_{n-1}}^Q - \mathcal{B}_{t_n}^Q$$

are independent and are Gaussian-distributed: for any $\phi \in L^2(\mathcal{D})$,

$$\langle \mathcal{B}_{t_{i+1}}^Q - \mathcal{B}_{t_i}^Q, \phi\rangle \sim \mathcal{N}(0, (t_{i+1} - t_i)\langle Q\phi, \phi\rangle)$$

for some bounded, non-negative, symmetric and linear operator Q

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$$\mathcal{B}_t^Q = \sum_{i\in\mathbb{N}} \sqrt{\lambda_k^Q} \beta_t^{(k)} e_k^Q, \quad t \in [0, T]$$

where $(\beta_t^{(k)})_{t\in[0,T]}$ are independent Brownian motions, $\{e_k^Q\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{D})$ consisting of eigenvectors of Q, with eigenvalues $\{\lambda_k^Q\}_{k\in\mathbb{N}}$:

$$Q e_k^Q = \lambda_k^Q e_k^Q$$

$\rightarrow$ "Karhunen-Loeve" decomposition

■ SPATIO-TEMPORAL SPDE

GEOLEARNING

- Space-time random field $(\mathcal{Z}(t,x))_{t\in[0,T],x\in\mathcal{D}}$ seen as a $L^2(\mathcal{D})$-valued stochastic process $(\mathcal{Z}_t)_{t\in[0,T]}$

- Consider the following SPDE / infinite dimensional SDE

$$\boxed{\mathrm{d}\mathcal{Z}_t + (\kappa^2 - \Delta)^{\alpha/2}\mathcal{Z}_t \mathrm{d}t = \mathrm{d}\mathcal{B}_t^Q}$$

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- White-noise formulation

$$\frac{\partial \mathcal{Z}}{\partial t} + (\kappa^2 - \Delta)^{\alpha/2}\mathcal{Z} = \mathcal{W}_T \otimes \mathcal{W}_S^Q$$

where $\mathcal{W}_T \otimes \mathcal{W}_S^Q$ is a space-time "white noise", linked to $\mathcal{B}_t^Q$ through

$$\forall t \in [0,T], \quad \langle \mathcal{B}_t, \phi_S\rangle - \langle \mathcal{B}_0, \phi_S\rangle = \langle \mathcal{W}_T \otimes \mathcal{W}_S^Q, \mathbf{1}_{[0,t]} \otimes \phi_S\rangle = \int_0^t \int_{\mathcal{D}} \phi_S(x) \mathcal{W}_T \otimes \mathcal{W}_S^Q(\mathrm{d}t, \mathrm{d}x)$$

DIFFUSION SPDE

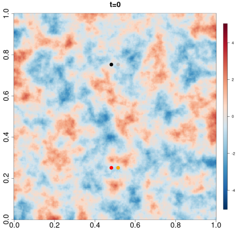
Let $\{e_k\}_{k\in\mathbb{N}}$ be an o.n.b. of $L^2(\mathcal{D})$ composed of eigenfunctions of $-\Delta$ with eigenvalalues $\{\lambda_k\}_{k\in\mathbb{N}}$

- Let $Q = (\kappa^2 - \Delta)^{-\alpha_\mathrm{n}}$ and $\kappa > 0, \alpha_\mathrm{d} \geq 0, \alpha_\mathrm{n} \geq 0$, and consider the diffusion SPDE

$$\boxed{\begin{cases} \mathrm{d}\mathcal{Z}_t + (\kappa^2 - \Delta)^{\alpha_\mathrm{d}}\ \mathcal{Z}_t\ \mathrm{d}t = \mathrm{d}\mathcal{B}_t^Q \\ \mathcal{Z}_0 = z_0 \in L^2(\mathcal{D}) \end{cases} \quad \text{with } \mathcal{B}_t^Q = \sum_{k\in\mathbb{N}} (\kappa^2 + \lambda_k)^{-\alpha_\mathrm{n}/2} \beta_t^{(k)} e_k^Q, \quad t \in [0, T]}$$

where $(\beta_t^{(k)})_{t\in[0,T]}$ are independent Brownian motions

$\rightarrow$ Existence and uniqueness of (weak and mild) solution if $\alpha_\mathrm{d} + \alpha_\mathrm{n} > d/2$



DIFFUSION SPDE

GEOLEARNING

Let $\{e_k\}_{k\in\mathbb{N}}$ be an o.n.b. of $L^2(\mathcal{D})$ composed of eigenfunctions of $-\Delta$ with eigenvalalues $\{\lambda_k\}_{k\in\mathbb{N}}$

- Consider the diffusion SPDE

$$\boxed{\begin{cases} \mathrm{d}\mathcal{Z}_t + (\kappa^2-\Delta)^{\alpha_\mathrm{d}}\ \mathcal{Z}_t\ \mathrm{d}t = \mathrm{d}\mathcal{B}_t^Q \\ \mathcal{Z}_0 = z_0 \in L^2(\mathcal{D}) \end{cases} \quad \text{with } \mathcal{B}_t^Q = \sum_{k\in\mathbb{N}} (\kappa^2+\lambda_k)^{-\alpha_\mathrm{n}/2}\beta_t^{(k)} e_k^Q, \quad t\in[0,T]}$$

- Spectral decomposition of the SPDE solution:

$$\mathcal{Z}_t = \sum_{k\in\mathbb{N}} Z_t^{(k)} e_k$$

where for each k, $Z^{(k)}$ satisfies a SDE

$$\begin{cases} \mathrm{d}Z_t^{(k)} + (\kappa^2+\lambda_k)^{\alpha_\mathrm{d}}\ Z_t^{(k)}\ \mathrm{d}t = (\kappa^2+\lambda_k)^{-\alpha_\mathrm{n}/2}\mathrm{d}\beta_t^{(k)}, \quad t\in(0,T] \\ Z_0^{(k)} = \langle z_0, e_k\rangle \end{cases}$$

$\rightarrow$ *Independent* Ornstein-Uhlenbeck process

$$\mathrm{d}Z_t^{(k)} = -\theta_k\ Z_t^{(k)}\ \mathrm{d}t + \tau_k\ \mathrm{d}\beta_t^{(k)}, \quad t\in(0,T]$$

DIFFUSION SPDE

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where for each k, $\theta_k = (\kappa^2 + \lambda_k)^{\alpha_\mathrm{d}}$, $\tau_k = (\kappa^2 + \lambda_k)^{-\alpha_\mathrm{n}/2}$ and

$$\mathrm{d}Z_t^{(k)} = -\theta_k \; Z_t^{(k)} \; \mathrm{d}t + \tau_k \; \mathrm{d}\beta_t^{(k)}$$

- Explicit dynamic for the coeffcients $Z_t^{(k)}$ $\rightarrow$ Gaussian + Markov process

$$(Z_{t+\delta t}^{(k)} | Z_t^{(k)} = z) \sim \mathcal{N}\Big(e^{-\theta_k \delta t} z, \; \frac{\tau_k^2}{2\theta_k}(1 - e^{-2\theta_k \delta t})\Big) \quad (\delta t > 0)$$

- Covariance of the (unconditionned) process

$$\mathrm{Cov}(Z_{t+\delta t}^{(k)}, Z_t^{(k)}) = \frac{\tau_k^2}{2\theta_k} e^{-\theta_k \delta t} \quad (\delta t > 0)$$

SOLVING THE DIFFUSION SPDE

- **Method 1 : Spectral Approach**

1. Pick a truncation order N
2. Simulate the OH processes $Z^{(1)}, \ldots, Z^{(N)}$ for $t \in (0, T]$
3. Return $\mathcal{Z}_t = \sum_{k=1}^{N} Z_t^{(k)} e_k$, $t \in (0, T]$

$\rightarrow$ Need to know the eigendecomposition of $-\Delta$....

SOLVING THE DIFFUSION SPDE

- **Method 2 : Euler+FEM** (Clarotto et al., 2022)

1. Time-discretization using (Implict) Euler

$$\mathcal{Z}_{t+\delta t} - \mathcal{Z}_t + (\kappa^2 - \Delta)^{\alpha_{\mathrm{d}}}\ \mathcal{Z}_{t+\delta t}\ \delta t = \int_t^{t+\delta t} \mathrm{d}\mathcal{B}_s^Q = \sqrt{\delta t}\mathcal{W}_S^Q$$

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2. Space disctretization using FEM (Galerkin) to deduce that, with $\widehat{\mathcal{Z}}_t = \sum_{i=1}^{N_h} [\boldsymbol{z}^{(t)}]_i \psi_i$,

$$\big(\boldsymbol{I} + \delta t(\kappa^2\boldsymbol{I} + \widetilde{\boldsymbol{R}})^{\alpha_\mathrm{d}}\big)\boldsymbol{C}^{1/2}\boldsymbol{z}^{(t+\delta t)} = \boldsymbol{C}^{1/2}\boldsymbol{z}^{(t)} + \sqrt{\delta t}\boldsymbol{C}^{1/2}(\kappa^2\boldsymbol{I} + \widetilde{\boldsymbol{R}})^{-\alpha_\mathrm{n}/2}\boldsymbol{w}$$

where $\boldsymbol{w} \sim \mathcal{N}(0, \boldsymbol{I})$, and $\boldsymbol{C}$ (diagonal mass lumped matrix), $\boldsymbol{R}$ (stiffness matrix) and $\widetilde{\boldsymbol{R}} = \boldsymbol{C}^{-1/2}\boldsymbol{R}\boldsymbol{C}^{-1/2}$ are sparse matrices

$\rightarrow$ Explicit expression for precision matrix of solution!

SOLVING THE DIFFUSION SPDE

- **Method 3 : Discrete spectral approach**

1. Formulate the spectral solution in the FEM space: $\widehat{\mathcal{Z}}_t = \sum_{k=1}^{N_h} \widehat{Z}_t^{(k)} E_k$ where

$$\begin{cases} \mathrm{d}\widehat{Z}_t^{(k)} + (\kappa^2 + \Lambda_k)^{\alpha_\mathrm{d}} \ \widehat{Z}_t^{(k)} \ \mathrm{d}t = (\kappa^2 + \Lambda_k)^{-\alpha_\mathrm{n}/2} \mathrm{d}\beta_t^{(k)}, \quad t \in (0, T] \\ \widehat{Z}_0^{(k)} = \langle z_0, E_k \rangle \end{cases}$$

SOLVING THE DIFFUSION SPDE

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2. Use the cond. dist. of OU processes + definition of $-\Delta_h$ to deduce that for any $\delta t > 0$

$$\widehat{\mathcal{Z}}_{t+\delta t} = m_{\delta t}(-\Delta_h)\widehat{\mathcal{Z}}_t + \sigma_{\delta t}(-\Delta_h)\widehat{\mathcal{W}},$$

where $\widehat{\mathcal{W}}$ is a white noise on the FEM space and $m_{\delta t}, \sigma_{\delta t}$ are given by

$$m_{\delta t}(\lambda) = e^{-\delta t(\kappa^2+\lambda)^{\alpha_{\mathrm{d}}}}, \quad \sigma_{\delta t}(\lambda) = (\sqrt{2})^{-1}(\kappa^2 + \lambda)^{-(\alpha_{\mathrm{d}}+\alpha_{\mathrm{n}})/2}\sqrt{1 - e^{-2(\kappa^2+\lambda)^{\alpha_{\mathrm{d}}}\delta t}}$$

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3. Use Galerkin-Chebyshev to compute the solution

$$\boldsymbol{C}^{1/2}\boldsymbol{z}^{(t+\delta t)} = m_{\delta t}(\widetilde{\boldsymbol{R}})\boldsymbol{C}^{1/2}\boldsymbol{z}^{(t)} + \sigma_{\delta t}(\widetilde{\boldsymbol{R}})\boldsymbol{w}$$

$\rightarrow$ Explicit expression for precision matrix of solution!

COVARIANCE OF THE SOLUTION

Recall that $\theta_k = (\kappa^2 + \lambda_k)^{\alpha_{\mathrm{d}}}$, $\tau_k = (\kappa^2 + \lambda_k)^{-\alpha_{\mathrm{n}}/2}$

- For $\phi, \varphi \in L^2(\mathcal{D})$, and $t, u \in \mathbb{R}$

$$\mathrm{Cov}\big(\langle \mathcal{Z}_t, \phi\rangle, \langle \mathcal{Z}_{t+u}, \varphi\rangle\big) = \sum_{k\in\mathbb{N}} \mathrm{Cov}\big(Z_t^{(k)}, Z_{t+u}^{(k)}\big)\langle e_k, \phi\rangle\langle e_k, \varphi\rangle = \sum_{k\in\mathbb{N}} \frac{\tau_k^2}{2\theta_k} e^{-\theta_k u}\langle e_k, \phi\rangle\langle e_k, \varphi\rangle$$

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- Spatial trace : $u = 0$

$$\mathrm{Cov}\big(\langle \mathcal{Z}_t, \phi\rangle, \langle \mathcal{Z}_t, \varphi\rangle\big) = \sum_{k\in\mathbb{N}} \frac{\tau_k^2}{2\theta_k}\langle e_k, \phi\rangle\langle e_k, \varphi\rangle = \sum_{k\in\mathbb{N}} \frac{1}{2}(\kappa^2 + \lambda_k)^{-(\alpha_\mathrm{d}+\alpha_\mathrm{n})}\langle e_k, \phi\rangle\langle e_k, \varphi\rangle$$

$\rightarrow$ Same as the solution of the SPDE $(\kappa^2 - \Delta)^{(\alpha_\mathrm{d}+\alpha_\mathrm{n})/2}\mathcal{Y} = \sqrt{2}\,\mathcal{W}$

COVARIANCE OF THE SOLUTION

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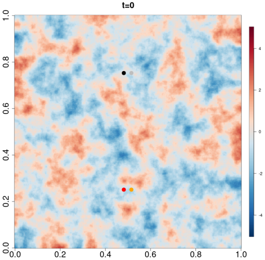
$\rightarrow$ Same as the solution of the SPDE $(\kappa^2 - \Delta)^{(\alpha_\mathrm{d}+\alpha_\mathrm{n})/2}\mathcal{Y} = \sqrt{2}\,\mathcal{W}$

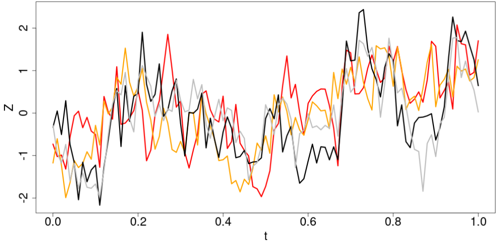
- Temporal trace : $\phi = \varphi$

$$\mathrm{Cor}\big(\langle \mathcal{Z}_t, \phi\rangle, \langle \mathcal{Z}_{t+u}, \phi\rangle\big) = \sum_{k\in\mathbb{N}} \frac{c_k}{\sum_{l\in\mathbb{N}} c_l} e^{-\theta_k u}, \quad c_k = (\kappa^2 + \lambda_k)^{-(\alpha_\mathrm{d}+\alpha_\mathrm{n})}\langle e_k, \phi\rangle^2$$

$\rightarrow$ Mixture of exponential correlation functions!

■ EXAMPLES OF SAMPLES





- Time regularity: Almost-nowhere differentiable, but paths are Hölder-continuous with order $1/2 - \varepsilon$ for any $\varepsilon \in (0, 1/2)$, i.e.

$$\mathbb{E}[|\langle \mathcal{Z}_t, \phi \rangle - \langle \mathcal{Z}_{t+u}, \phi \rangle|] \leq \mathrm{C}|u|^{1/2-\varepsilon} \tag{1}$$

$\rightarrow$ Regularity inherited from the Brownian motion! Can we do better?

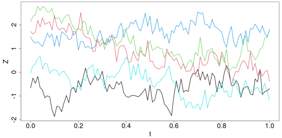
OUTLINE

GEOLEARNING

FRACTIONAL BROWNIAN MOTION

- A fractional Brownian motion $(\beta_t^H)_{t\in[0,T]}$ with Hurst index $H \in (0,1)$ is a Gaussian process such that
  - $(\beta_t^H)_{t\in[0,T]}$ has (almost-surely) continuous trajectories
  - Its covariance function is given by

$$\mathrm{Cov}(\beta_t^H, \beta_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$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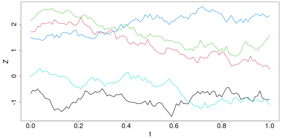
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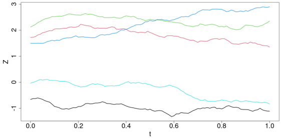
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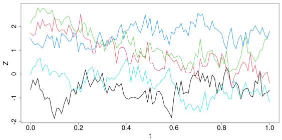
$$\mathrm{Cov}(\beta_t^H, \beta_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

- Some properties
  - Particular case : $H = 1/2 \rightarrow$ (standard) Brownian motion
  - Associated variogram: $\mathrm{Var}(\beta_t^H - \beta_s^H) \propto |t-s|^{2H}$
  - Paths Hölder-continuous of order $H-\varepsilon$ for any $\varepsilon \in (0,H)$
  - Stationary increments: $\beta_t^H - \beta_s^H \sim \beta_{t-s}^H$
  - Increments are positively correlated if $H > 1/2$ ($\rightarrow$ tends to continue the trends) and negatively correlated if $H < 1/2$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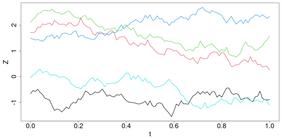
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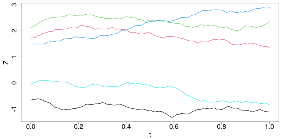
H=0.75

FRACTIONAL BROWNIAN MOTION

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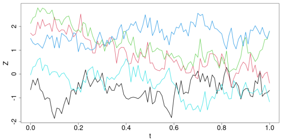
  - Its covariance function is given by

$$\mathrm{Cov}(\beta_t^H, \beta_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

- Note: if $H \neq 1/2$, the increments are not independent $\rightarrow$ Non Markovian process (with long-range dependence for $H > 1/2$)

- Remark: Fractional Brownian motion can also be seen as an integrated (standard) Brownian motion

- (Fractional) Itô calculus and (fractional) Q-Wiener processes are well-defined for fractional Brownian motions



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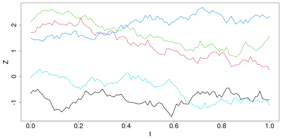

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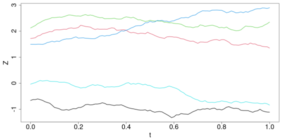

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where $(\beta_t^{(H,k)})_{t\in[0,T]}$ are independent fractional Brownian motions

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$$\mathrm{d}Z_t^{(k)} = -\theta_k\ Z_t^{(k)}\ \mathrm{d}t + \tau_k\ \mathrm{d}\beta_t^{(H,k)}, \quad t \in (0,T]$$

■ COVARIANCE OF THE STATIONARY SOLUTION

Recall that $\theta_k = (\kappa^2 + \lambda_k)^{\alpha_{\rm d}}$, $\tau_k = (\kappa^2 + \lambda_k)^{-\alpha_{\rm n}/2}$

- For $\phi, \varphi \in L^2(\mathcal{D})$, and $t, u \in \mathbb{R}$

$$\mathrm{Cov}\big(\langle \mathcal{Z}_t, \phi\rangle, \langle \mathcal{Z}_{t+u}, \varphi\rangle\big) = \sum_{k\in\mathbb{N}} C_k(u)\langle e_k, \phi\rangle\langle e_k, \varphi\rangle$$

where $C_k(u) = \mathrm{Cov}\big(Z_t^{(k)}, Z_{t+u}^{(k)}\big) = \dfrac{\Gamma(2H+1)\sin(\pi H)}{\pi}\dfrac{\tau_k^2}{2\theta_k^{2H}}\displaystyle\int_{-\infty}^{\infty} e^{ix(u/\theta_k)}\frac{|x|^{1-2H}}{1+x^2}\mathrm{d}x$

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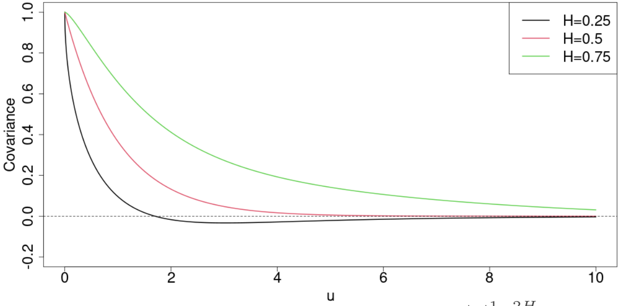

Covariance function $\rho(u) = \int_{-\infty}^{\infty} e^{ix(u/\theta_k)} \frac{|x|^{1-2H}}{1+x^2}\mathrm{d}x$

COMPUTING THE SOLUTION

- Remark: The noise increments are now time-correlated
- **Method : Discrete spectral + Implicit Euler time-stepping**

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$$\widehat{\mathcal{Z}}_t = \sum_{k=1}^{N_h} \widehat{Z}_t^{(k)} E_k \quad \text{where} \begin{cases} \mathrm{d}\widehat{Z}_t^{(k)} + (\kappa^2 + \Lambda_k)^{\alpha_\mathrm{d}} \ \widehat{Z}_t^{(k)} \ \mathrm{d}t = (\kappa^2 + \Lambda_k)^{-\alpha_\mathrm{n}/2} \mathrm{d}\beta_t^{(k,H)} \\ \widehat{Z}_0^{(k)} = \langle z_0, E_k \rangle \end{cases}$$

COMPUTING THE SOLUTION

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2. Generate independent fBm trajectories $\boldsymbol{\beta}_t^{(h)} = (\beta_t^{(1,H)}, \ldots, \beta_t^{(N_h,H)})$ over $[0, T]$

COMPUTING THE SOLUTION

GEOLEARNING

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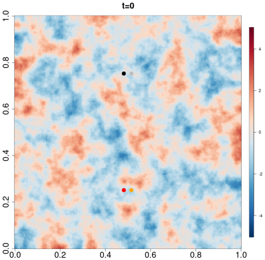
3. Use Implicit time step on SDEs + Change of basis to deduce that, with $\widehat{\mathcal{Z}}_t = \sum_{i=1}^{N_h} [\boldsymbol{z}^{(t)}]_i \psi_i$,

$$\text{where } \big(\boldsymbol{I} + \delta t(\kappa^2 \boldsymbol{I} + \widetilde{\boldsymbol{R}})^{\alpha_\mathrm{d}}\big) \boldsymbol{C}^{1/2} \boldsymbol{z}^{(t+\delta t)} = \boldsymbol{C}^{1/2} \boldsymbol{z}^{(t)} + \sqrt{\delta t} \boldsymbol{C}^{1/2} (\kappa^2 \boldsymbol{I} + \widetilde{\boldsymbol{R}})^{-\alpha_\mathrm{n}/2} \big(\boldsymbol{\beta}_{t+\delta t}^{(h)} - \boldsymbol{\beta}_t^{(h)}\big)$$

$\rightarrow$ Explicit expression for precision matrix of solution with Kronecker products!

■ EXAMPLES OF SAMPLES



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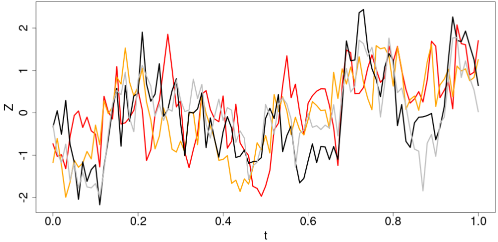

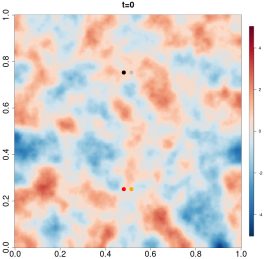

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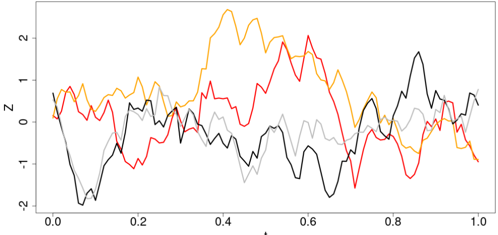
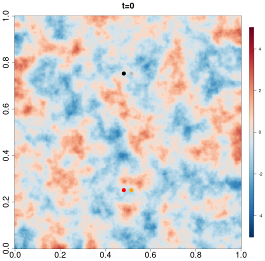
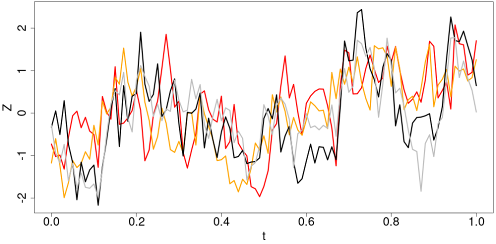


■ EXAMPLES OF SAMPLES

GEOLEARNING

H=0.5





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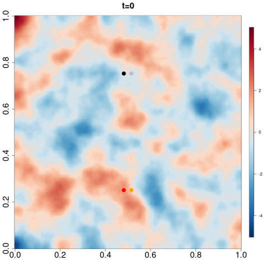
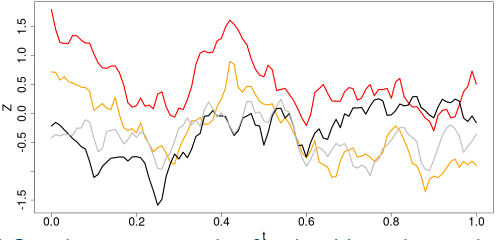
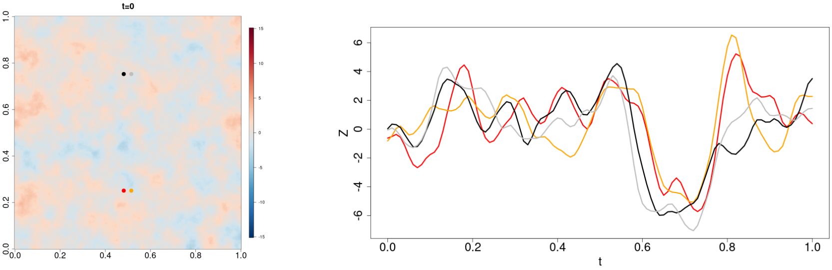




OUTLOOKS

- Stochastic analysis of numerical scheme (ongoing)
- Link to fractional operators
- Going further: What if we replaced the fBm with even smoother processes?

Matern process instead of fBm



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