Extreme temporal modelling

Concomitant extremes

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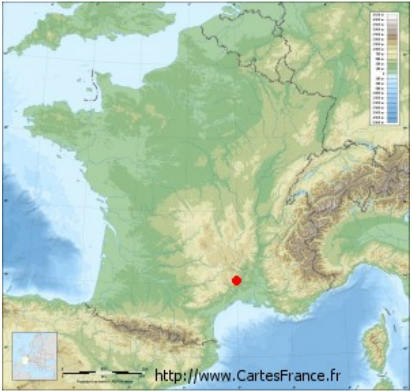
Motivation



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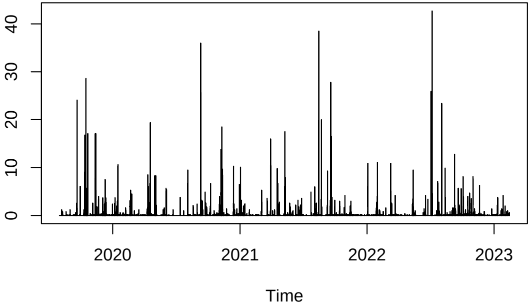

Figure: Hourly precipitation measurements (in mm) at Générargues in Occitanie region from August 2019 to February 2023

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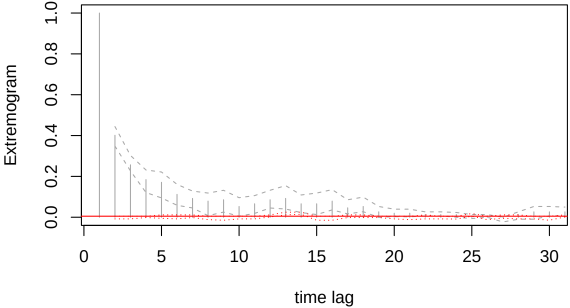

Figure: Extremogram: $\lim_{x\to\infty} \mathbb{P}(X_t > x \mid X_0 > x)$

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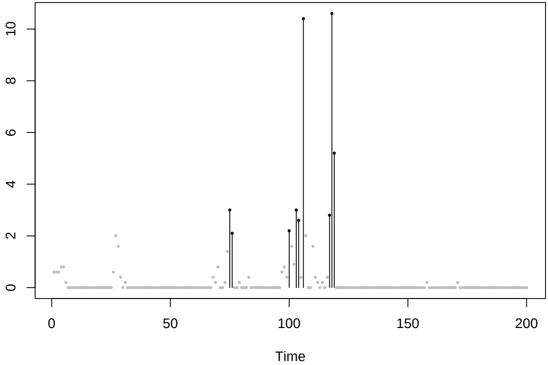

Figure: Sample extract of hourly precipitation measurements (in mm) at Générargues in Occitanie region from August 2019 to February 2023

Goal

- Model extremal temporal clustering of stationary time series.
- Infer cluster statistics using block methods.

Key words: Extreme value theory, cluster Poisson point process, blocks methods for cluster inference, extremal index.

Notation

We consider

- $(\mathbf{X}_t)$ stationary time series in $(\mathbb{R}^d, |\cdot|)$.

[1] Janssen (2019)

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- $(\mathbf{X}_t)$ is regularly varying (RV_α): for all $h \geqslant 0$, $y > 1$,

$$\lim_{x \to +\infty} \mathbb{P}\Big(|\mathbf{X}_0| > y\,x, \frac{\mathbf{X}_{[-h,h]}}{|\mathbf{X}_0|} \in \cdot \,\Big|\, |\mathbf{X}_0| > x\Big) \;=\; y^{-\alpha}\, \mathbb{P}(\Theta_{[-h,h]} \in \cdot).$$

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- $|\mathbf{\Theta}_t| \to 0$ as $t \to +\infty$ a.s.
- $\mathbf{Q} = \mathbf{\Theta}/\|\mathbf{\Theta}\|_\alpha$, where $\|\mathbf{x}\|_\alpha^\alpha = \sum_{t\in\mathbb{Z}} |\mathbf{x}_t|^\alpha$. [1]

[1] Janssen (2019)

Cluster Poisson Point Process

Theorem[2] Buriticá, Meyer, Mikosch, Wintenberger (2021)

Assume $(\mathbf{X}_t)$ satisfies $\mathbf{RV}_\alpha$, **AC** and **MX**, then

$$N_n = \sum_{i=1}^{n} \varepsilon_{a_n^{-1} X_i} \quad \to \quad N = \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \varepsilon_{\Gamma_i^{-1/\alpha} \mathbf{Q}_{ij}},$$

in $\mathbb{R}_0$, where $n\mathbb{P}(|\mathbf{X}_1| > a_n) \to 1$,

• (Γ_i) are points of a unit rate homogeneous Poisson process on $(0, \infty)$: $\Gamma_1 \le \Gamma_2 \le \cdots$.

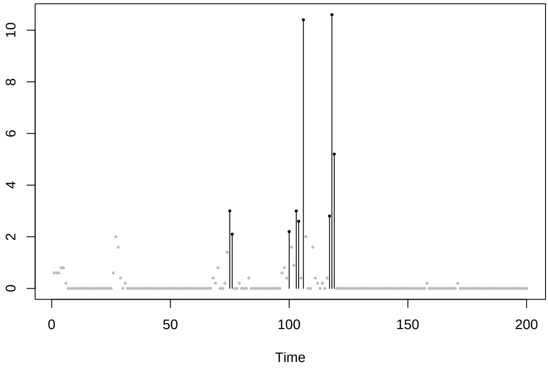
• $\sum_{j \in \mathbb{Z}} \varepsilon_{\mathbf{Q}_{ij}}$, $i = 1, 2, \ldots$, is an iid sequence of point processes with state space $\mathbb{R}^d$ with generic element $\mathbf{Q}_i = (\mathbf{Q}_{ij})_{j \in \mathbb{Z}}$,

$$\mathbf{Q} = \left(\frac{\Theta_j}{\|\Theta\|_\alpha} \right)_{j \in \mathbb{Z}}.$$

• (Γ_i) and $(\mathbf{Q}_i)_{i=1,2,\ldots}$ are independent.

[2] Davis and Hsing (1995)

Point Process Illustration



i.i.d. model

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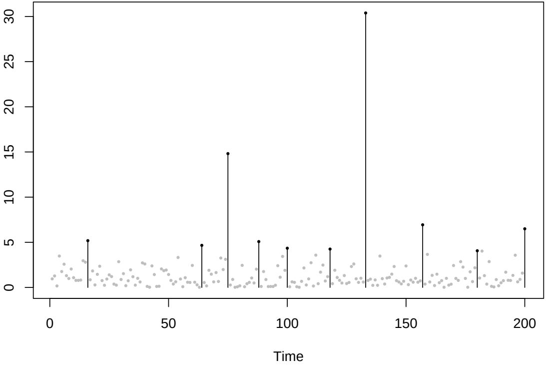

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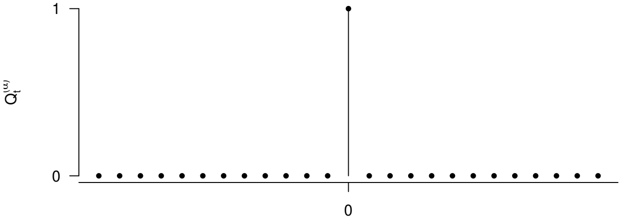
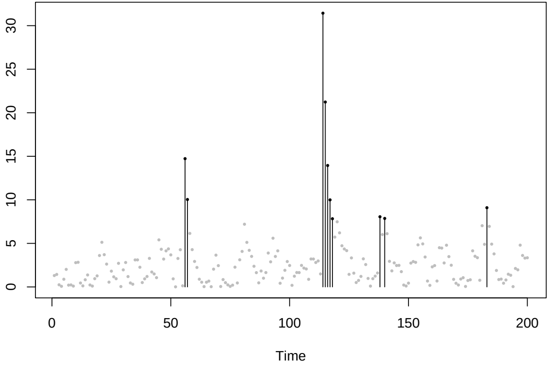


Auto-regressive model

- (X_t) a stationary **AR(1)**, $X_t = \varphi X_{t-1} + Z_t$ with $\varphi \in (0,1)$, and (Z_t) i.i.d. satisfying $\mathbf{RV}_\alpha$,



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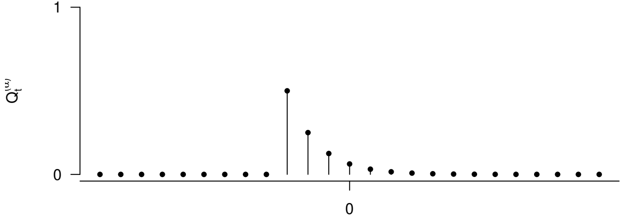
$$Q_t^{(\alpha)} \quad = \quad \Theta_t/\|\Theta\|_\alpha \quad = \quad \varphi^t \Theta_0^Z \, \mathbb{1}(J + t \geqslant 0)\,(1-\varphi^\alpha)^{1/\alpha},$$

J independent of Θ_0^Z, $\mathbb{P}(J = j) = (1-\varphi^\alpha)\varphi^{j\alpha}, j \geqslant 0.$



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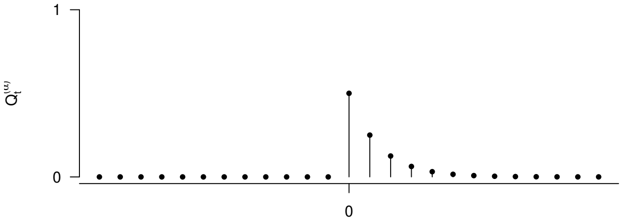
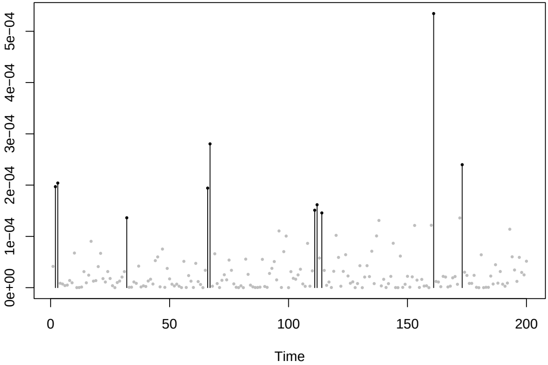


Causal solution to SRE under Kesten-Goldie assumptions

- (X_t) **causal solution to SRE**, $X_t = A_t X_{t-1} + B_t$, $((A_t, B_t))$ positive i.i.d. and $((A, B))$ satisfies Kesten-Goldie theory then



We take $A_t = e^{N_t - 1/2}$ such that (N_t) is i.i.d. gaussian noise, and we follow Example 6.1. in Janßen and Segers (2014) where $\Theta_{-t} = A_{-t} \cdots A_{-1}$, for $t \leqslant 0$.

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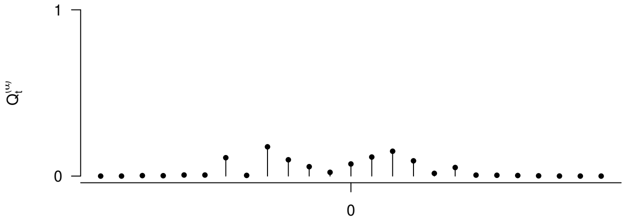

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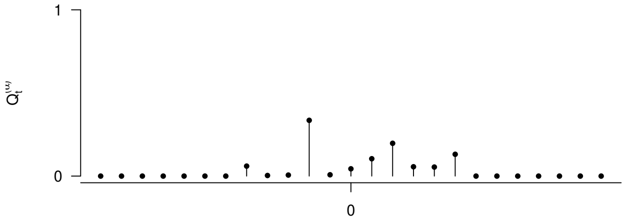

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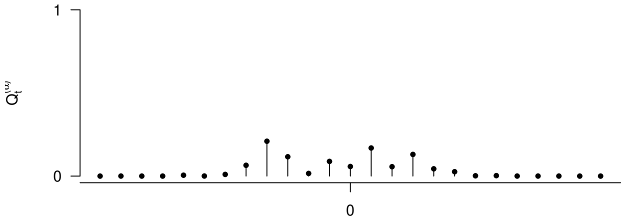

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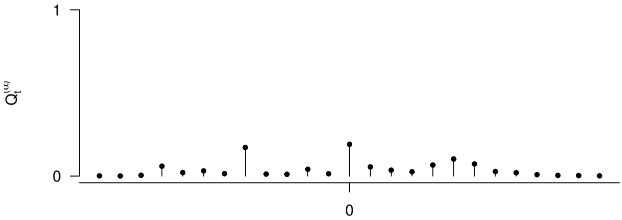

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Cluster inference

Blocks method

Aim: compute cluster statistic: $\mathbb{E}[f(\mathbf{Q})]$.

So far: $N_n = \sum_{i=1}^{n} \varepsilon_{a_n^{-1} X_i} \to N = \sum_{i=1}^{\infty} \sum_{j \in \mathbb{Z}} \varepsilon_{\Gamma_i^{-1/\alpha} \mathbf{Q}_{ij}}$.

$$\mathbf{X}_{[1:n]} = \big(\underbrace{\mathbf{X}_{[1:b_n]}}_{:=\mathcal{B}_1}, \underbrace{\mathbf{X}_{b_n+[1:b_n]}}_{:=\mathcal{B}_2}, \ldots, \underbrace{\mathbf{X}_{[n-b_n+1:b_n]}}_{:=\mathcal{B}_{m_n}} \big),$$

- select k extremal blocks $\mathcal{B}_{(1)}, \ldots, \mathcal{B}_{(k)}$,
- average $\frac{1}{k} \sum_{t=1}^{k} f(\mathcal{B}_{(t)}/a_n)$,
- e.g. count threshold exceedances in a block $f : (\mathbf{x}_t) \mapsto \sum \mathbb{1}(|\mathbf{x}_t| > 1)$.

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(Q) How to choose those k extremal blocks ?

Large deviations of ℓ^α-blocks

Theorem Buriticá, Mikosch, Wintenberger (2023)
Assume $(\mathbf{X}_t)$ satisfies $\mathbf{RV}_\alpha$, and (x_n) satisfies $\mathbf{AC}(x_n)$, $\mathbf{CS}_\alpha(x_n)$, and $n\mathbb{P}(|\mathbf{X}_1| > x_{b_n}) \to 0$. Then,

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and

$$\lim_{n\to\infty} \mathbb{P}(\|\mathbf{X}_{[1,b_n]}\|_\alpha > x_{b_n})/(b_n\mathbb{P}(|\mathbf{X}_0| > x_{b_n})) \quad = \quad 1.$$

Asymptotic normality

We propose to estimate the statistic $f_\alpha^{\mathbf{Q}} = \mathbb{E}[f_\alpha(Y\mathbf{Q})]$ by

$$\widehat{f_\alpha^{\mathbf{Q}}} := \frac{1}{k}\sum_{t=1}^{m} f_{\widehat{\alpha}}\big(\mathcal{B}_t/\|\mathcal{B}_t\|_{\widehat{\alpha},(k+1)}\big)\mathbb{1}(\|\mathcal{B}_t\|_{\widehat{\alpha}} > \|\mathcal{B}_t\|_{\widehat{\alpha},(k+1)}),$$

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Theorem Buriticá, Wintenberger (2024)

Under moment, bias and mixing conditions. There exists $k = k_n \to \infty$, $m/k \to \infty$, such that for suitable $f_\alpha : \ell^\alpha \to \mathbb{R}$.

$$\sqrt{k_n}\left(\widehat{f_{\widehat{\alpha}}^{\mathbf{Q}}} - f_\alpha^{\mathbf{Q}}\right) \xrightarrow{d} \mathcal{N}\Big(0, Var\big(f_\alpha(Y\mathbf{Q})\big)\Big), \quad n \to \infty,$$

and $k_n/k_n' \to 0$, Y independent of $\mathbf{Q}$, and $\mathbb{P}(Y > y) = y^{-\alpha}$, for $y > 1$.

Tail index Hill estimator

$$\frac{1}{\widehat{\alpha}^n} := \frac{1}{\widehat{\alpha}^n(k')} := \frac{1}{k'} \sum_{t=1}^{n} \log(|\mathbf{X}_t|/|\mathbf{X}|_{(k'+1)}),$$

where $|\mathbf{X}|_{(1)} \geqslant |\mathbf{X}|_{(2)} \geqslant \cdots \geqslant |\mathbf{X}|_{(n)}$, and $k' = k'(n)$ is a tuning sequence for the Hill estimator satisfying $k' \to \infty$, $n/k' \to \infty$, as $n \to \infty$.

Extremal index

Maximum domain of attraction

There exists (a_n) such that

$$(\mathbb{P}(|\mathbf{X}_1| \leqslant x a_n))^n \quad \to \quad G(x) := \mathbb{P}((\Gamma_1)^{-1/\alpha} \leqslant x), \quad n \to \infty,$$

where $G(x) = \exp\{-x^{-\alpha}\}$, for $\alpha > 0$, $x > 1$, $n\mathbb{P}(X_1 > a_n) \to 1$.

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(Leadbetter 1983) there exists $\theta \in (0, 1]$ such that

$$\mathbb{P}(\|\mathbf{X}_{[1,n]}\|_\infty \leqslant x a_n) \quad \to \quad (G(x))^{\theta}, \quad n \to \infty.$$

Example: extremal index

Cluster-based extremal index inference

For example, if $f_\alpha : (\mathbf{x}_t) \mapsto \|(\mathbf{x}_t)\|_\infty^\alpha / \|(\mathbf{x}_t)\|_\alpha^\alpha$, then,

$$\theta_{|\mathbf{X}|} \quad = \quad \mathbb{E}[\|\mathbf{Q}\|_\infty^\alpha].$$

$\implies$ Estimator of the extremal index based on extremal ℓ^α-blocks.

$$\widehat{\theta}_{|\mathbf{X}|} \quad = \quad \frac{1}{k} \sum_{t=1}^{m} \frac{\|\mathcal{B}_t\|_\infty^{\widehat{\alpha}}}{\|\mathcal{B}_t\|_{\widehat{\alpha}}^{\widehat{\alpha}}} \mathbb{1}(\|\mathcal{B}_t\|_{\widehat{\alpha}} > \|\mathcal{B}\|_{\widehat{\alpha},(k+1)}),$$

Application: Générargues

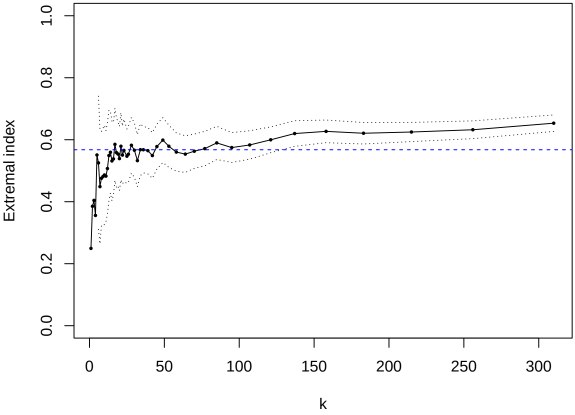

Figure: Extremal index estimate hourly precipitation measurements (in mm) in Générargues for an estimate $\hat{\alpha} = 1.807 \pm 0.2$

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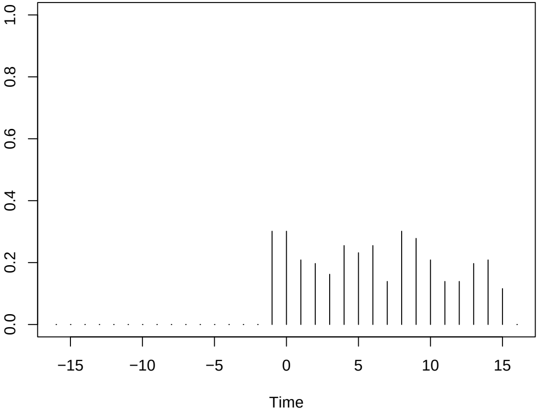

Figure: Empirical estimates of the cluster process Q.

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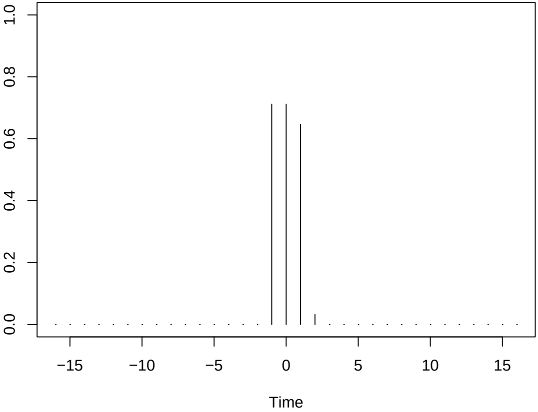

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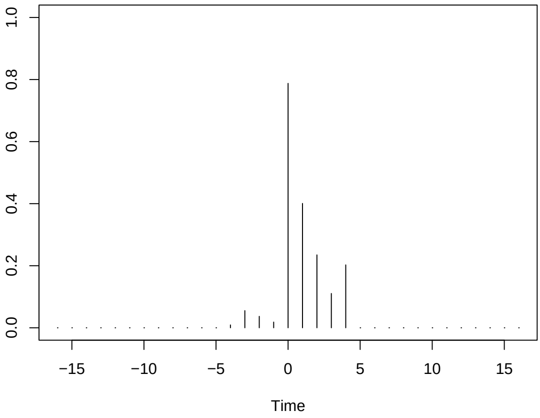

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Conclusions

- How to define/aggregate extreme observations in time?
- $\ell^{\hat{\alpha}}$-blocks inference yields robust result.
- Toolbox to assess statistical properties of temporal extremes.