Statistical modeling of spatio-temporal data distributed over surfaces

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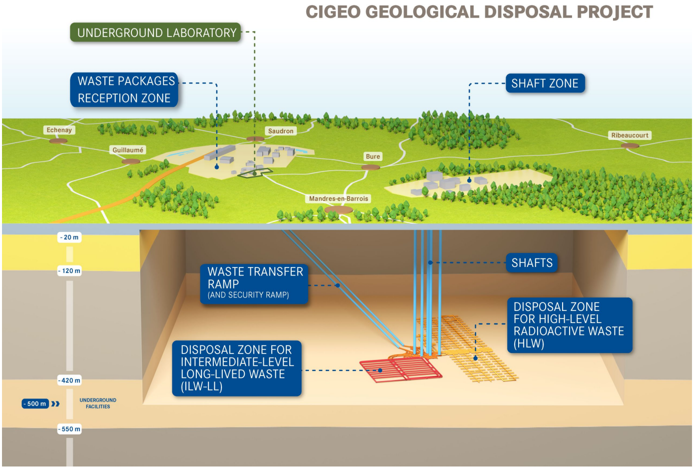
Content

Cigeo project



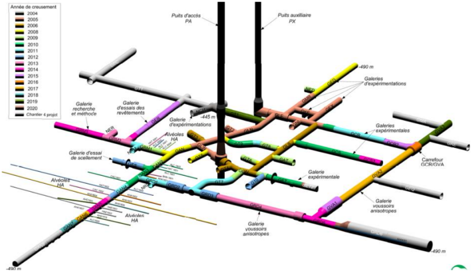
Objectives

Predict the evolution of the gallery during its operational phase based on measurements taken by sensors

2 quantities of interest:

- Temperature
- Deformation

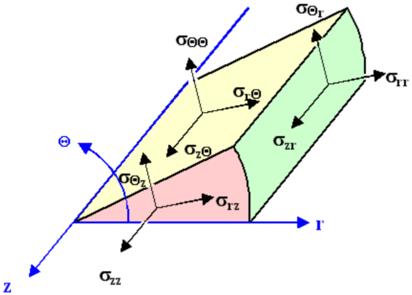
Multiple sensors in the zone of interest, provide data everyday since 2017



Deformations

The sensors collects measurements of the 3 components of the deformation:

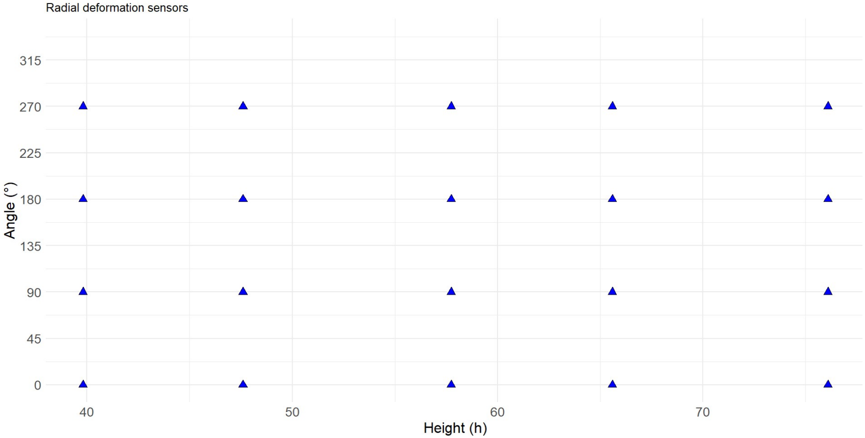
- Radial deformation
- Orthoradial deformation
- Longitudinal deformation



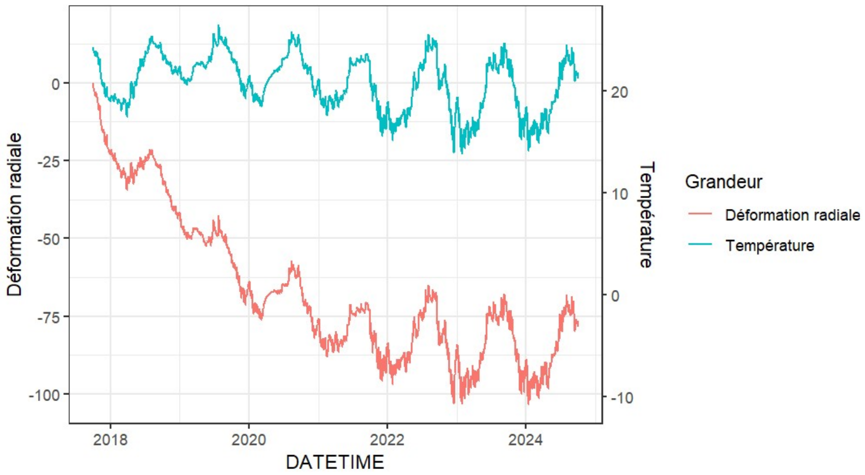
Here, focus and radial deformation and temperature

Sensors position

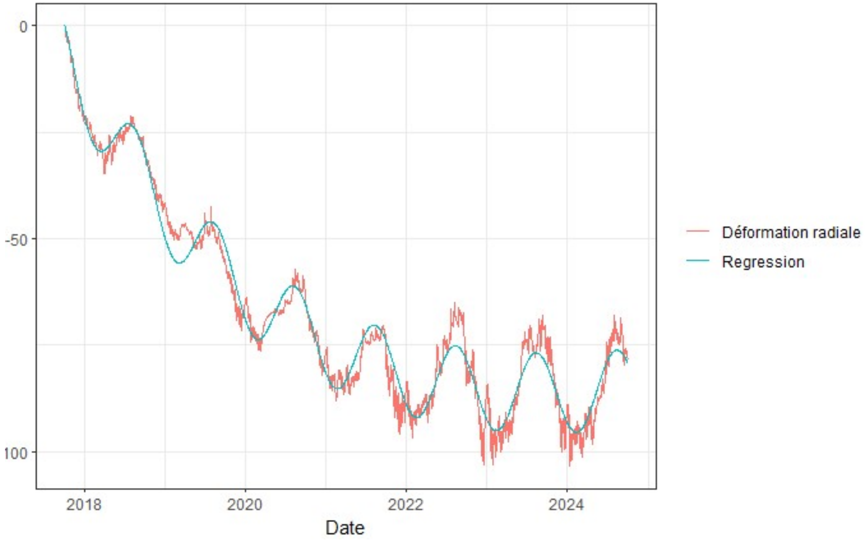
One deformation sensor ⇔ One temperature sensor



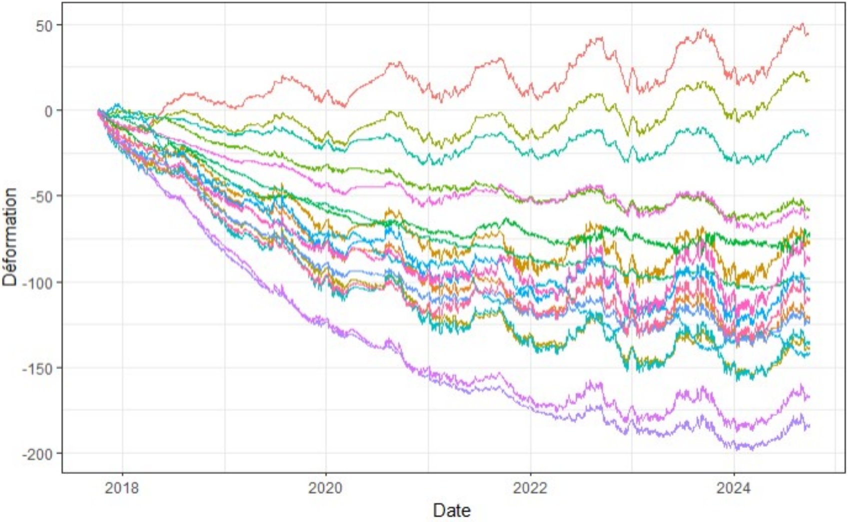
Signals


Déformation radiale
0
-25
-50
-75
-100
2018
2020
2022
2024
DATETIME
20
10
0
-10
Température
Grandeur
Déformation radiale
Température

Trend estimation



Different trends



Trend modeling for deformation

Every trend can be expressed

$$\mu_{\text{Defo}}(t,\mathbf{s}) = \beta_0(\mathbf{s}) + \beta_1(\mathbf{s}) exp\left(-\beta_2(\mathbf{s})t\right) + \beta_3(\mathbf{s})\sin\left(\frac{2\pi}{365}t + \beta_4(\mathbf{s})\right)$$

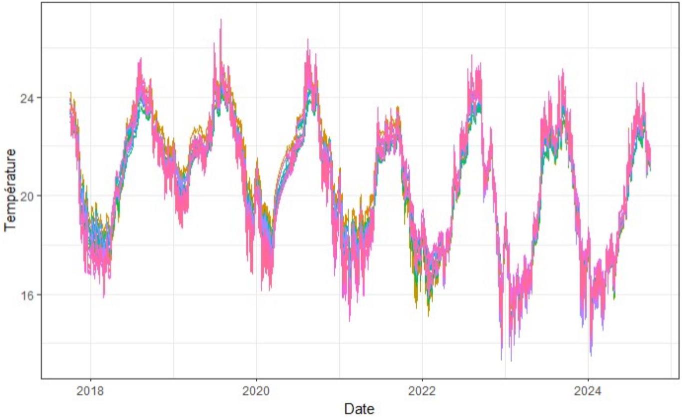
Problem: Seek for linear relation in $\boldsymbol{\beta}(\mathbf{s}) = (\beta_0(\mathbf{s}), \beta_1(\mathbf{s}), \beta_2(\mathbf{s}), \beta_3(\mathbf{s}), \beta_4(\mathbf{s}))$

Idea: Express trend as

$$\begin{aligned}\mu_{\text{Defo}}(t,\mathbf{s}) = \beta_0(\mathbf{s}) & \\ & + \beta_1(\mathbf{s})\exp\left(-c_1 t\right) + \beta_2(\mathbf{s})\exp\left(-c_2 t\right) \\ & + \beta_3(\mathbf{s})\sin\left(\frac{2\pi}{365}t + c_3\right) + \beta_4(\mathbf{s})\sin\left(\frac{2\pi}{365}t + c_4\right)\end{aligned}$$

where c_1, c_2, c_3 and c_4 do not depend on $\mathbf{s}$

Trend modeling for temperature



$$\mu_{\text{Temp}}(t, \mathbf{s}) = \alpha_0(\mathbf{s}) + \alpha_1(\mathbf{s}) \sin\left(\frac{2\pi}{365} t + c_5\right)$$

Estimating the coefficients

Question : How to define functions $\beta_j(\mathbf{s})$ and $\alpha_j(\mathbf{s})$

Idea :

- Denote $(\mathbf{s}_i)_{i=1}^n$ the positions of the sensors
- Compute $\beta_j(\mathbf{s}_i)$ and $\alpha_j(\mathbf{s}_i),\ 1 \leq i \leq n$ with linear regression
- Use interpolation method to get $\beta_j(\mathbf{s})$ and $\alpha_j(\mathbf{s})$ on the surface of the cylinder

Specific study: Work with interpolating splines on Riemannian manifold

Content

Splines in $\mathbb{R}$

Considering points $(x_i)_{i=1}^n$ in $\mathbb{R}$ and observations $(y_i)_{i=1}^n$ at these points

The interpolation spline problem is to minimize

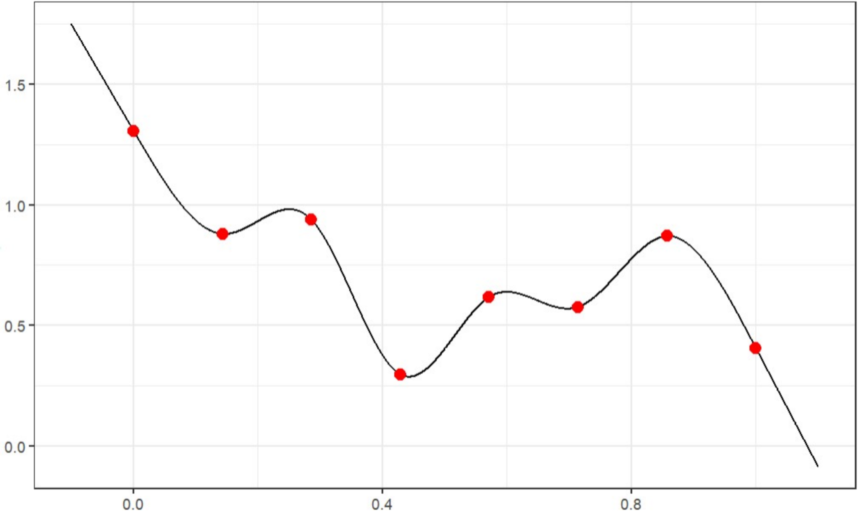
$$E(u) = \int_{\mathbb{R}} \left[u''(x)\right]^2 dx$$

given the constraint $u(x_i) = y_i, \ 1 \leq i \leq n$

Solution: $u^\star(x) = a_0 + a_1 x + \sum_{i=1}^n b_i |x - x_i|^3$ [1]
with

$$\begin{cases} \sum_i b_i = 0 \\ \sum_i b_i x_i = 0 \\ u(x_i) = y_i \quad \text{for } 1 \leq i \leq n. \end{cases}$$

Splines 1D



Interpretation

Why the form $u^\star(x) = a_0 + a_1 x + \sum_{i=1}^n b_i |x - x_i|^3$?

- $x \mapsto a_0 + a_1 x$: Space of function with null energy E
- $\sum_{i=1}^n b_i |x - x_i|^3 = \sum_{i=1}^n b_i K(x_i - x)$ with $K(h) = |h|^3$

Where this Kernel comes from ?

Fourier integral of a function u :

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) e^{i\omega x} d\omega, \ \hat{u} \text{ the Fourier transform of } u.$$

Functions $\phi_\omega(x) = e^{i\omega x}$ the eigenvectors of the second derivative operator.

Fourier transform of $K(h) \propto \frac{1}{\omega^4}$, and ω^2 is the eigenvalue associated to $\phi_\omega(x)$

Generalization on Riemannian manifold

- $\mathcal{M}$ a $d-$dimensional compact Riemannian manifold
- Δ the Laplace-Beltrami operator on $\mathcal{M}$

$(\phi_k)_k$ eigen functions of Δ is a complete orthonormal basis for $L^2(\mathcal{M})$.
Eigvenvalues λ_k with $\lambda_0 = 0$ and $\lambda_k \leq \lambda_{k+1}$

- $\forall u \in L^2(\mathcal{M}), u = \sum_k u_k \phi_k$
- $\forall u \in L^2(\mathcal{M}), \Delta u = \sum_k \lambda_k u_k \phi_k$

We define

$$H(\mathcal{M}) = H_0(\mathcal{M}) \oplus H_1(\mathcal{M}), \quad \text{where}$$

- $H_0(\mathcal{M}) = \mathcal{E}_0$
- $H_1(\mathcal{M}) = \bigoplus_{k \neq 0} \mathcal{E}_k$
- $\mathcal{E}_k$ eigenspace associated with λ_k

Generalization on Riemannian manifold

$H_0(\mathcal{M}), H_1(\mathcal{M})$ and $H(\mathcal{M})$ are RKHS with scalar products

- $\langle u, v \rangle_0 = u_0 \bar{v_0}$
- $\langle u, v \rangle_1 = \sum_{k \neq 0} \lambda_k^2 u_k \bar{v_k}$
- $\langle u, v \rangle = \langle u, v \rangle_0 + \langle u, v \rangle_1$

and reproducing kernels

- $K_0(\mathbf{s}_1, \mathbf{s}_2) = \phi_0(\mathbf{s}_1)\bar{\phi_0}(\mathbf{s}_2)$
- $K_1(\mathbf{s}_1, \mathbf{s}_2) = \sum_{k \neq 0} \frac{1}{\lambda_k^2} \phi_k(\mathbf{s}_1)\bar{\phi_k}(\mathbf{s}_2)$
- $K(\mathbf{s}_1, \mathbf{s}_2) = K_0(\mathbf{s}_1, \mathbf{s}_2) + K_1(\mathbf{s}_1, \mathbf{s}_2)$

Generalization on Riemannian manifold

Interpolation problem is to minimize the norm in $H_1(\mathcal{M})$:

$$\|u\|_{H_1(\mathcal{M})} = \sum_k \lambda_k^2 |u_k|^2 = \int_M |\Delta u|^2$$

subject to the constraint $u(\mathbf{s}_i) = y_i$, with $s_i \in \mathcal{M}, 1 \leq i \leq n$,

Solution [3]

$$u^\star(\mathbf{s}) = a_0 \Phi_0(\mathbf{s}) + \sum_{i=1}^{n} b_i K_1(\mathbf{s}, \mathbf{s}_i) \text{ with}$$

- $\forall 1 \leq j \leq n, \sum_{i=1}^{n} b_i K_1(s_j, s_i) + a_0 \Phi_0(\mathbf{s}_i) = y_j$
- $\sum_{i=1}^{n} b_i \Phi_0(\mathbf{s}_i) = 0$

Analogy with case 1D

When $\mathcal{X} = \mathcal{M}$, compact manifold we have

- $u = \sum_k u_k \phi_k$
- $\Delta u = \sum_k \lambda_k u_k \phi_k$
- $H_0 = \{a_0 \phi_0\}$
- $K_1(\mathbf{s}_1, \mathbf{s}_2) = \sum_{k \neq 0} \frac{1}{\lambda_k^2} \phi_k(\mathbf{s}_1) \bar{\phi_k}(\mathbf{s}_2)$

When $\mathcal{X} = \mathbb{R}$, we have

- $u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) e^{i\omega x} d\omega$
- $u''(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} w^2 \hat{u}(\omega) e^{i\omega x} d\omega$
- $H_0 = \{x \mapsto a_0 + a_1 x\}$
- $K_1(x_1, x_2) \propto \int_{\mathbb{R}} \frac{1}{\omega^4} e^{-i\omega(x_2 - x_1)} d\omega$ inverse Fourier transform of $\omega \mapsto \frac{1}{\omega^4}$

Splines vs Gaussian process regression

Challenge: The eigenvalues and eigenvectors (λ_k, ϕ_k) are unknwon

Idea: Equivalence between Splines interpolation and GP Regression [2]

The solution $u^\star$ of the interpolation problem is

$$u^\star(\mathbf{s}) = \mathbb{E}\left(Y(\mathbf{s}) \mid y(\mathbf{s}_1), \ldots, y(\mathbf{s}_n)\right)$$

where $Y(\mathbf{s})$ is a Gaussian process of

- Prior mean $a_0\phi_0(\mathbf{s})$, with unknown a_0
- Covariance kernel K_1

Can we simulate $Y(\mathbf{s})$ on Riemannian manifold ?

Expansion on $(\phi_k)_k$

Consider centered GRF Z through the expansion [5]

$$Z = \sum_k f^{1/2}(\lambda_k) W_k \phi_k,$$

with $f^{1/2}$ such that $(f^{1/2})^2 = f$ and $(W_k)_k$ independant Gaussian variables

Z has covariance function

$$C(\mathbf{s}_1, \mathbf{s}_2) = \sum_k f(\lambda_k)\phi_k(\mathbf{s}_1)\phi_k(\mathbf{s}_2)$$

Then, here we define

$$f(\lambda) = \begin{cases} 0, & \text{if } \lambda = 0 \\ \frac{1}{\lambda^2}, & \text{else.} \end{cases}$$

Galerkin approximation

Consider triangulation T of $\mathcal{M}$:

- m nodes $(\mathbf{c}_1, \ldots, \mathbf{c}_m)$
- Compact support functions $\psi_1, \ldots, \psi_m$: ψ_i piecewise linear equal to 1 at $\mathbf{c}_i$ and 0 at all other nodes to define approximation

$$Z_{\mathrm{T}}(\mathbf{s}) = \sum_{i=1}^{m} Z(\mathbf{c}_i)\psi_i(\mathbf{s})$$

Objective :

$$Z_{\mathrm{T}}(\mathbf{s}) = \sum_{k=1}^{m} f^{1/2}(\lambda_k^{(m)}) W_k \phi_k^{(m)}$$

with $\lambda_k^{(m)}$ and $\phi_k^{(m)}$ the eigenvalues and eigenvectors of the discretization Δ_m of the Laplace-Beltrami operator.

Galerkin approximation

- Mass matrix $\mathbf{M} = [\langle \psi_i, \psi_j \rangle]$ and stiffness matrix $\mathbf{G} = [\langle \nabla\psi_i, \nabla\psi_j \rangle]$
- Matrix $\mathbf{S} = (\sqrt{\mathbf{M}})^{-1}\mathbf{G}(\sqrt{\mathbf{M}})^{-T}$ with $\mathbf{M}$ such that $\sqrt{\mathbf{M}}(\sqrt{\mathbf{M}})^T = \mathbf{M}$

Then, $(\lambda_k^{(m)})_{k=1}^m$ are the eigenvalues of $\mathbf{S}$

From [4], $\mathbf{S}$ is diagonalizable:

$$\mathbf{S} = \mathbf{V}\mathrm{Diag}(\lambda_1^{(m)}, \ldots, \lambda_m^{(m)})\mathbf{V}^T$$

Galerkin approximation

$$Z_{\mathrm{T}} = \sum_{k=1}^{m} f^{1/2}(\lambda_k^{(m)}) W_k \phi_k^{(m)} = \sum_{i=1}^{m} Z_i \psi_i$$

with $\mathbf{Z} = (Z_1, \ldots, Z_n)$ centered Gaussian vector with covariance

$$\mathbf{\Sigma} = (\sqrt{\mathbf{M}})^{-T} f(\mathbf{S}) (\sqrt{\mathbf{M}})^{-1}$$

with

$$f(\mathbf{S}) = \mathbf{V}\mathrm{Diag}(f(\lambda_1^{(m)}), \ldots, f(\lambda_m^{(m)}))\mathbf{V}^T$$

Galerkin approximation

$\Rightarrow Z_{\mathrm{T}}$ is a centered GRF with covariance kernel

$$K_1^{(m)}(\mathbf{s}_1, \mathbf{s}_2) = \sum_{k=1}^{m} f(\lambda_k^{(m)})\phi_k^{(m)}(\mathbf{s}_1)\phi_k^{(m)}(\mathbf{s}_2)$$

and $Z_{\mathrm{T}}(\mathbf{s}) = \sum_{i=1}^{m} Z_i\psi_i = \mathbf{A}(\mathbf{s})\mathbf{Z}$ with $\mathbf{A}(\mathbf{s}) = (\psi_1(\mathbf{s}), \ldots, \psi_m(\mathbf{s}))$

Then

$$K_1^{(m)}(\mathbf{s}_1, \mathbf{s}_2) = \mathbf{A}(\mathbf{s}_1)\mathbf{\Sigma}\mathbf{A}(\mathbf{s}_2)^T$$

Summary

1. We have observations $\boldsymbol{y} = (y_i)_{i=1}^n$ at points $(\mathbf{s}_i)_{i=1}^n$ on a manifold $\mathcal{M}$
2. Build triangulation of $\mathcal{M}$
3. Consider $Z_{\mathrm{T}}(\mathbf{s}) = \sum_{i=1}^m Z_i \psi_i = \mathbf{A}(\mathbf{s})\mathbf{Z}$ with $\mathbf{Z} = (Z_1, \dots, Z_m)$ of covariance $\boldsymbol{\Sigma} = (\sqrt{\mathbf{M}})^{-T} f(\mathbf{S})(\sqrt{\mathbf{M}})^{-1}$
4. Get $\phi_1^{(m)}$ eigenvector of Δ_m associated with $\lambda_1^{(m)} = 0$: $\phi_1^{(m)} = \sum_{k=1}^m \left[(\sqrt{M})^{-T}\mathbf{v}_1\right]_k \psi_k$ with $\mathbf{v}_1$ first column of $\mathbf{V}$
5. Consider the GRF $Y_{\mathrm{T}}(\mathbf{s}) = a_0 \phi_1^{(m)}(\mathbf{s}) + Z_{\mathrm{T}}(\mathbf{s})$ where

$$a_0 = (\mathbf{P}^T \mathbf{K}^{-1} \mathbf{P})^{-1} \mathbf{P}^T \mathbf{K}^{-1} \boldsymbol{y} \text{ where}$$

$$\mathbf{P} = (\phi_1^{(m)}(\mathbf{s}_1), \dots, \phi_1^{(m)}(\mathbf{s}_n))^T,\ \mathbf{K} = \mathbf{A}_n \boldsymbol{\Sigma} (\mathbf{A}_n)^T,\ \mathbf{A}_n = (\mathbf{A}(\mathbf{s}_j))_{j=1}^n$$

Interpolation

Back to our interpolation problem

$$u^\star(\mathbf{s}) = \mathbb{E}\left(Y_\mathrm{T}(\mathbf{s}) \mid Y_\mathrm{T}(\mathbf{s}_1) = y_1, \ldots, Y_\mathrm{T}(\mathbf{s}_n) = y_n\right)$$

and then with the classical posterior expectation formula

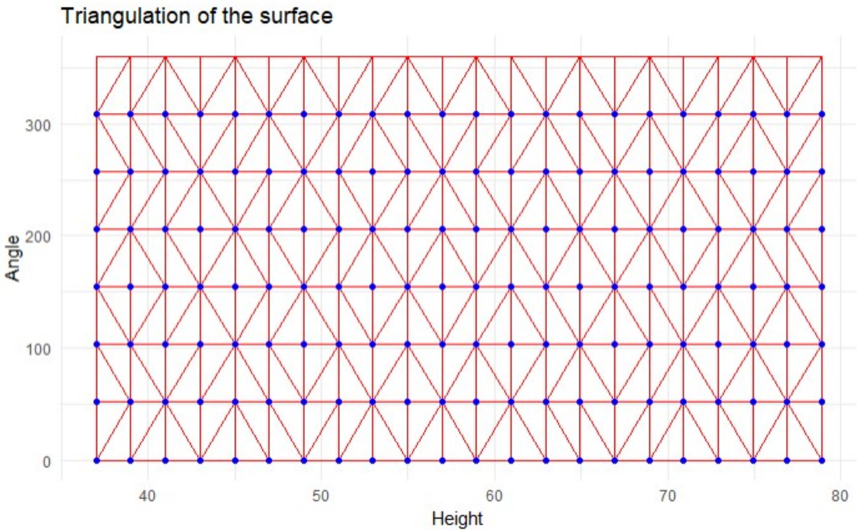
$$u^\star(\mathbf{s}) = a_0\phi_1^{(m)}(\mathbf{s}) + \mathbf{A}(\mathbf{s})\mathbf{\Sigma}(\mathbf{A}_n)^T\left(\mathbf{A}_n\mathbf{\Sigma}(\mathbf{A}_n)^T\right)^{-1}(\boldsymbol{y} - a_0\mathbf{P})$$

with

- $a_0 = (\mathbf{P}^T\mathbf{K}^{-1}\mathbf{P})^{-1}\mathbf{P}^T\mathbf{K}^{-1}\boldsymbol{y}$
- $\mathbf{P} = (\phi_1^{(m)}(\mathbf{s}_1), \ldots, \phi_1^{(m)}(\mathbf{s}_n))^T$
- $\mathbf{A}(\mathbf{s}) = (\psi_1(\mathbf{s}), \ldots, \psi_m(\mathbf{s}))$ and $\mathbf{A}_n = (\mathbf{A}(\mathbf{s}_j))_{j=1}^n$

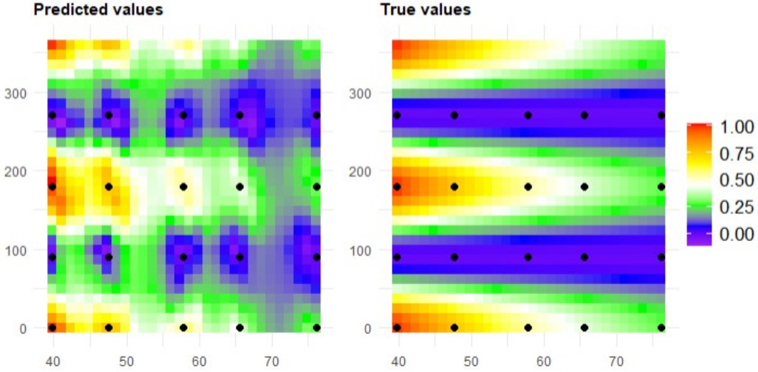
Content

Triangulation

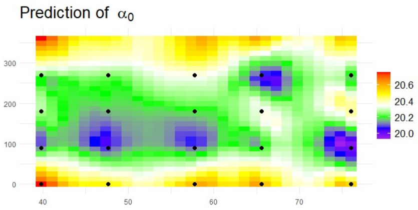


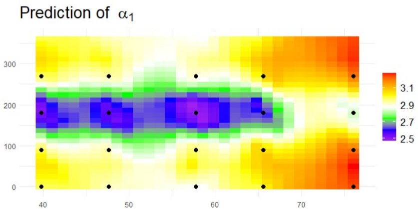
Analytical test case

Consider function $f(\theta, h) = \cos(\theta)^2 \exp\left(-\frac{h-h_{\min}}{30}\right)$

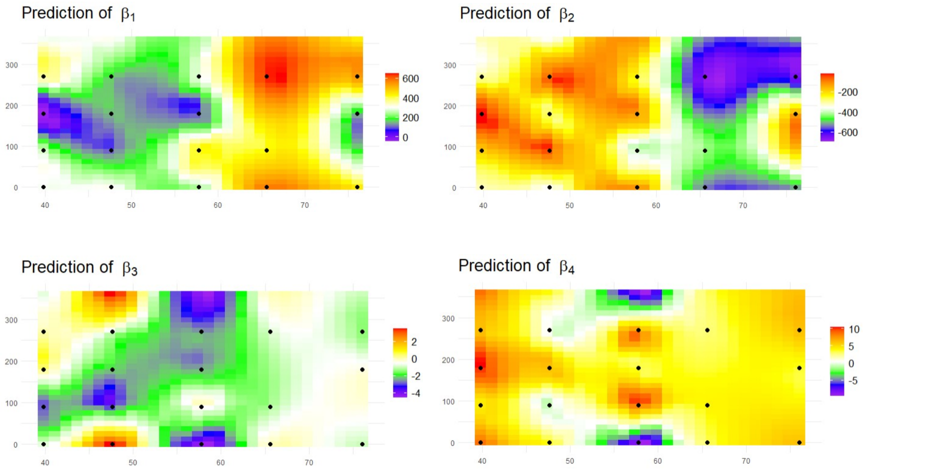


Coefficients for temperature





Coefficients for deformation



Content

Future work

Problem: For the moment, need to diagonalize $\mathbf{S}$ to work with

$$f(\mathbf{S}) = \mathbf{V}\mathrm{Diag}(f(\lambda_1^{(m)}), \ldots, f(\lambda_m^{(m)}))\mathbf{V}^T$$

Impossible in practice if very fine mesh

Question: How to work directly with precision matrix with singular $\mathbf{\Sigma}$?

$\Rightarrow$ Adapt work on intrinsic GMRF [6]

Content

Spatio-temporal model

We consider the model

$$\begin{cases} Y_{\text{Defo}}(t, \mathbf{s}) = \mu_{\text{Defo}}(t, \mathbf{s}) + X_{\text{Defo}}(t, \mathbf{s}) + E_{\text{Defo}}(t, \mathbf{s}) \\ Y_{\text{Temp}}(t, \mathbf{s}) = \mu_{\text{Temp}}(t, \mathbf{s}) + X_{\text{Temp}}(t, \mathbf{s}) + E_{\text{Temp}}(t, \mathbf{s}) \end{cases}$$

- $\mu(t, \mathbf{s})$ the deterministic trend trend
- $X(t, \mathbf{s})$ the centered random residual
- $E(t, \mathbf{s})$ the measurement white noise

Question: How to predict $X_{\text{Defo}}(t, \mathbf{s})$ and $X_{\text{Temp}}(t, \mathbf{s})$

Account for dependency

We consider

$$\begin{bmatrix} X_{\text{Temp}}(\mathbf{s}, t) \\ X_{\text{Defo}}(\mathbf{s}, t) \end{bmatrix} = \mathbf{B} \begin{bmatrix} X_1(\mathbf{s}, t) \\ X_2(\mathbf{s}, t) \end{bmatrix}$$

with

- $\mathbf{B} = \begin{bmatrix} 1 & b_1 \\ b_2 & 1 \end{bmatrix}$
- $X_1(t, \mathbf{s})$ and $X_2(t, \mathbf{s})$ independent spatio-temporal random fields solution of SPDE

Discretization:

$$\begin{cases} X_1^{\mathrm{T}}(\mathbf{s}, t) = \sum_{i=1}^{m} X_1^i(t)\psi_i(\mathbf{s})) \\ X_2^{\mathrm{T}}(\mathbf{s}, t) = \sum_{i=1}^{m} X_2^i(t)\psi_i(\mathbf{s}) \end{cases}$$

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